

Example: Section 6.4

$$\vec{x}' = P\vec{x}, \quad P = \begin{pmatrix} -k & 0 & 0 \\ 0 & i & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \vec{x}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad k > 0$$

* Eigenvalues:

$$|P - I| = \begin{vmatrix} -k-1 & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -1 \end{vmatrix} = -(k+1) \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} = -(k+1)(1^2 + 1)$$

$$\Rightarrow |P - I| = -(k - (-i))(1 - i)(1 + i)$$

Three eigenvalues: $-k, i, -i$

We already know that the matrix P is not defective.

* Eigenvectors:

$$\text{L1: } A + kI = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -k & -1 \\ 0 & 1 & k \end{pmatrix} \quad \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{L2: } A - iI = \begin{pmatrix} -k-i & 0 & 0 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \sim \begin{pmatrix} -k-i & 0 & 0 \\ 0 & i & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \begin{array}{l} x_1 = 0 \\ x_2 + x_3 = 0 \end{array}$$

$$\text{L2: } \begin{pmatrix} 0+i & 0 \\ 0 & 0 \end{pmatrix} \quad \Rightarrow \vec{v}_2 = \begin{pmatrix} 0 \\ i \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{b}}$$

* General form of the solution:

$$\vec{x}(t) = C_1 e^{-kt} \vec{v}_1 + C_2 e^{it} (\cos(\beta t) \vec{a} - \sin(\beta t) \vec{b})$$

$$+ C_3 e^{it} (\sin(\beta t) \vec{a} + \cos(\beta t) \vec{b})$$

$$= C_1 e^{-kt} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 \left(\cos t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \sin t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right) + C_3 \left(\sin t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \cos t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right)$$

* "Attempt" of a phase portrait

$$\vec{x}(t) = \underbrace{C_1 e^{-kt} \vec{v}_1}_{\text{going to } 0 \text{ in the direction of } \vec{v}_1} + \underbrace{C_2 (\cos(\vec{b}t) \vec{a} - \sin(\vec{b}t) \vec{b}) + C_3 (\sin(\vec{b}t) \vec{a} + \cos(\vec{b}t) \vec{b})}_{\text{"center" portrait}}$$

going to 0
in the direction of \vec{v}_1

~~"center"~~ "center" portrait

$$\vec{x}(t) = \begin{pmatrix} C_1 e^{-kt} \\ -C_2 \sin t + C_3 \cos t \\ C_2 \cos t + C_3 \sin t \end{pmatrix}$$

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} C_1 \\ C_3 \\ C_2 \end{pmatrix} : C_1 = C_3 = 1, C_2 = 0$$

$$\begin{aligned} \vec{x}(t) &= e^{-kt} \vec{v}_1 + (\sin(t) \vec{a} + \cos(t) \vec{b}) \\ &= \begin{pmatrix} e^{-kt} \\ \cos t \\ \sin t \end{pmatrix}. \end{aligned}$$

$$A\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} -k \\ 0 \\ 1 \end{pmatrix} \xrightarrow{\text{tangent at time 0}}$$

